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CONTACT PROBLEM FOR A STAMP WITH NARROW RECTANGULAR BASE

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The problem of impression of a stamp with narrow rectangular base into an elastic isotropic half-space under the effect of a vertical force is considered. This problem has been studied in [1, 2]. Asymptotic properties of the integral equation obtained, which goes over into a singular integral in the limit as the beam width diminishes permitting substantiation of the known Zimmerman-Winkler hypothesis, were established in [1]. An approximate solution of the integral equation from [1] was given in [2]. A brief survey of the research devoted to the problem of impressing a rectangular stamp is contained in [2, 3]. A more complete method of solving this problem is proposed below.

1. Let us consider a stamp in the shape of a narrow rectangle of length $2a$ and width 2δ, where $\varepsilon = \delta / a \ll 1$. Let a vertical force P impress this stamp into an elastic isotropic half-space $z \geqslant 0$. The force P passes through the center of gravity of the stamp and is directed along the z-axis.

Applying a two-dimensional Fourier integral transform to the Lamé equilibrium equations in rectangular xyz coordinates, we find

$$w(x, y, 0) = -\frac{1-\nu^2}{\pi E} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\alpha^2 + \beta^2)^{-1/2} \sigma_z^{**}(\alpha, \beta, 0)\, e^{-i(\alpha x + \beta y)}\, d\alpha\, d\beta \qquad (1.1)$$

Here E, ν are the Young's modulus and the Poisson's ratio of the material of the elastic half-space, respectively, w is the projection of the displacement vector on the z-axis, σ_z^{**} is the two-dimensional Fourier tranform of the normal stress σ_z. Formula (1.1) is valid under the condition of no shear stresses on the half-space boundary (at $z = 0$). This formula establishes the connection between vertical displacements of the half-space boundary and normal stresses at the boundary.

We set

$$p(x, y) = -\sigma_z(x, y, 0) \tag{1.2}$$

Let us turn to the problem of the stamp and assume that no friction occurs between the stamp and the half-space, and that there is no load on the half-space outside the stamp. To simplify the problem, let us also assume that the base of the stamp is flat.

In conformity with the hypothesis in [1], we assume

$$p(x, y) = \frac{p(x)}{\pi \sqrt{\delta^2 - y^2}} \qquad (|x| < a, \ |y| < \delta) \tag{1.3}$$

($p(x)$ is the pressure per unit length of the stamp).

Let us find the two-dimensional Fourier transform of the function $p(x, y)$. We have

$$p^{**}(\alpha, \beta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x, y) e^{i(\alpha x + \beta y)} \, dx \, dy.$$

Substituting the expression for $p(x, y)$ from (1.3) here and integrating, we find

$$p^{**}(\alpha, \beta) = \left(\frac{1}{2\pi}\right)^{1/2} J_0(\delta\beta) \, p^*(\alpha) \tag{1.4}$$

where $J_n(x)$ is the Bessel function of the first kind, $p^*(\alpha)$ is the one-dimensional Fourier transform of the function $p(x)$. Using (1.1), (1.2), (1.4) we have

$$w(x) = \frac{1 - \nu^2}{\pi E} \left(\frac{2}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} p^*(\alpha) A(x) e^{-i\alpha x} d\alpha$$

$$A(\alpha) = \int_0^{\infty} (\alpha^2 + \beta^2)^{-1/2} J_0(\delta\beta) \, d\beta, \quad w(x) \equiv w(x, 0, 0)$$

It is known [4] that $A(\alpha) = I_0(^1/_2 \delta |\alpha|) K_0(^1/_2 \delta |\alpha|)$, where $I_0(x)$, $K_0(x)$ are modified Bessel functions of the first and second kinds, respectively. In the case under consideration, $w(x)$ and $p(x)$ are even functions. Hence we obtain

$$p(x) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^{\infty} p^*(\alpha) \cos x\alpha \, d\alpha. \tag{1.5}$$

$$w(x) = \frac{2(1 - \nu^2)}{\pi E} \left(\frac{2}{\pi}\right)^{1/2} \int_0^{\infty} p^*(\alpha) I_0(^1/_2 \delta\alpha) K_0(^1/_2 \delta\alpha) \cos x\alpha \, d\alpha$$

2. Taking into account that

$$w(x) = c \quad \text{for } |x| < a, \qquad p(x) = 0 \quad \text{for } |x| > a$$

and using the relationship (1.5), we arrive at the dual integral equations

$$\int_0^{\infty} F(\xi) I_0\left(\frac{1}{2}\varepsilon\xi\right) K_0\left(\frac{1}{2}\varepsilon\xi\right) \cos t\xi \, d\xi = b, \quad 0 \leqslant t < 1 \tag{2.1}$$

$$\int_0^{\infty} F(\xi) \cos t\xi \, d\xi = 0, \quad 1 < t < \infty$$

$$F(\xi) = p^*\left(\frac{\xi}{a}\right), \quad b = \left(\frac{\pi}{2}\right)^{1/2}\frac{Eac}{1-\nu^2}, \quad \varepsilon = \frac{\delta}{a}, \quad t = \frac{x}{a}, \quad \xi = a\alpha \tag{2.2}$$

Here c is the quantity by which the stamp is impressed into the elastic half-space under the effect of the force P.

We seek the solution of (2.1) in the form

$$F(\xi) = b\sum_{n=0}^{\infty}(-1)^n A_{2n} J_{2n}(\xi) \tag{2.3}$$

It is known [4] that

$$\int_0^\infty J_{2n}(\xi)\cos t\xi\, d\xi = (-1)^n \frac{T_{2n}(t)}{\sqrt{1-t^2}} \times \begin{cases} 1, & 0 < t < 1 \\ 0, & 1 < t < \infty \end{cases} \tag{2.4}$$

($T_n(x)$ are Chebyshev polynomials of the first kind).

Substituting (2.3) into the second equation in (2.1) and taking account of (2.4) we see that the second equation in (2.1) is satisfied. Then substituting (2.3) into the first equation in (2.1), we obtain

$$\sum_{n=0}^{\infty}(-1)^n A_{2n}\int_0^\infty J_{2n}(\xi)\, I_0\left(\frac{1}{2}\varepsilon\xi\right) K_0\left(\frac{1}{2}\varepsilon\xi\right)\cos t\xi\, d\xi = 1, \quad 0 \leqslant t < 1 \tag{2.5}$$

Furthermore, we have

$$\cos t\xi = J_0(\xi) + 2\sum_{m=1}^{\infty}(-1)^m T_{2m}(t) J_{2m}(\xi), \quad 0 < t < 1 \tag{2.6}$$

Substituting (2.6) into (2.5) we arrive at the relationship

$$\sum_{n=0}^{\infty} A_{2n} C_{0n} + 2\sum_{n=0}^{\infty} A_{2n}\sum_{m=1}^{\infty}(-1)^m C_{mn} T_{2m}(t) = 1, \quad 0 < t < 1 \tag{2.7}$$

Here

$$C_{mn} = (-1)^n \int_0^\infty J_{2m}(\xi) J_{2n}(\xi)\, I_0\left(\frac{1}{2}\varepsilon\xi\right) K_0\left(\frac{1}{2}\varepsilon\xi\right) d\xi \tag{2.8}$$

$$m, n = 0, 1, 2, \ldots$$

It is necessary to expand the right side in (2.7) in Chebyshev polynomials also. We have

$$1 = b_0 T_0(t) + \sum_{m=1}^{\infty} b_2 T_{2m}(t) \tag{2.9}$$

$$b_0 = 1, \quad b_{2m} = 0 \quad m = 1, 2, \ldots, \qquad T_0(t) = 1$$

Comparing coefficients of the expansions in (2.7) and (2.9), we obtain

$$\sum_{n=0}^{\infty} C_{mn} A_{2n} = \delta_{m0}, \qquad m = 0, 1, 2, \ldots \tag{2.10}$$

(δ_{mn} is the Kronecker symbol).

The system (2.10) is an infinite system of linear equations for the expansion coefficients A_{2n}. In the general case, the solution of the system (2.10) can only be performed approximately by cutting it off at $m = n = N$ and calculating the first $N + 1$

coefficients of A_{2n} from the finite system obtained.

3. A formula can now be obtained for $p(x)$. On the basis of (1.5), (2.2) and (2.3) we have

$$p(x)=\left(\frac{2}{\pi}\right)^{1/2}\frac{b}{a}\sum_{n=0}^{\infty}(-1)^n A_{2n}\int_0^{\infty}J_{2n}(\xi)\cos\left(\frac{x\xi}{a}\right)d\xi$$

Taking account of (2.2) and (2.4), we finally obtain

$$p(x)=\frac{\pi Ec}{2(1-\nu^2)}\left(1-\frac{x^2}{a^2}\right)^{-1/2}\sum_{n=0}^{\infty}A_{2n}T_{2n}\left(\frac{x}{a}\right)\times\begin{cases}1 & \text{for } |x|<a\\ 0 & \text{for } |x|>a\end{cases} \tag{3.1}$$

The force acting on the stamp is

$$P=\int_{-a}^{a}dx\int_{-\delta}^{\delta}p(x,y)\,dy$$

Substituting the expression for $p(x, y)$ from (1.3) here, taking account of (3.1) and integrating, we find

$$c=\gamma\frac{(1-\nu^2)P}{Ea},\qquad \gamma=\frac{2}{\pi^2 A_0} \tag{3.2}$$

The depth of impression of the stamp can be determined from (3.2). On the basis of (3.1) and (3.2) we have

$$p(x)=\frac{P}{\pi a}\left(1-\frac{x^2}{a^2}\right)^{-1/2}\sum_{n=0}^{\infty}\frac{A_{2n}}{A_0}T_{2n}\left(\frac{x}{a}\right),\quad |x|<a \tag{3.3}$$

Therefore, the quantity c and the function $p(x)$ are determined by (3.2), (3.3). The coefficients A_{2n} $(n=0, 1, \ldots)$, which can be found from the system (2.10) enter into these formulas.

The system (2.10) can be represented as

$$A_{2m}=\delta_{m0}-\sum_{n=0}^{\infty}(C_{mn}-\delta_{mn})A_{2n}$$

This system can be solved by iteration by assuming

$$A_{2m}^{(r+1)}=\delta_{m0}-\sum_{n=0}^{\infty}(C_{mn}-\delta_{mn})A_{2n}^{(r)} \tag{3.4}$$

where $A_{2m}^{(r)}$ is the r-th approximation. The relationship

$$A_{2m}^{(r+1)}=\frac{1}{C_{mm}}\left[\delta_{m0}-\sum_{n=0}^{\infty}(1-\delta_{mn})C_{mn}A_{2n}^{(r)}\right]$$

yields a certain modification of (3.4) which has definite advantages.

4. Let us reduce (2.8) for the coefficients C_{mn} to a form more convenient for execution of calculations. It is known [5] that

$$J_n(z)=\frac{1}{\pi}\int_0^{\pi}\cos(n\theta-z\sin\theta)\,d\theta \tag{4.1}$$

$$J_m(z)J_n(z)=\frac{2}{\pi}\int_0^{\pi/2}J_{m+n}(2z\cos\theta)\cos[(m-n)\theta]\,d\theta,\quad \operatorname{Re}(m+n)>-1$$

After some manipulation, the first of formulas (4.1) can be written as

$$J_{2n}(z)=\frac{2}{\pi}\int_0^{\pi/2}\cos 2n\theta\cos(z\sin\theta)\,d\theta \tag{4.2}$$

Taking (4.1) and (4.2) into account we find

$$J_{2m}(\xi)J_{2n}(\xi)=\frac{4}{\pi^2}\int_0^{\pi/2}\cos[2(m-n)\varphi]\,d\varphi\int_0^{\pi/2}\cos[2(m+n)\theta]\cos[2\xi\cos\varphi\sin\theta]d\theta$$

Substituting this expression into (2.8) and integrating with respect to ξ, we finally obtain

$$C_{mn}=\frac{4}{\pi^2}(-1)^n\int_0^{\pi/2}\cos[2(m-n)\varphi]\,d\varphi\int_0^{\pi/2}\frac{\cos[2(m+n)\theta]}{g(\varphi,\theta;\varepsilon)}\,\mathrm{K}\left[\frac{\varepsilon}{g(\varphi,\theta;\varepsilon)}\right]d\theta$$

$$g(\varphi,\theta;\varepsilon)=(4\cos^2\varphi\sin^2\theta+\varepsilon^2)^{1/2},\quad m,\ n=0,1,2,\ldots \tag{4.3}$$

($\mathrm{K}(k)$ is the complete elliptic integral of the first kind).

The coefficients C_{mn} can now be calculated by using an electronic digital computer by replacing the integrals in (4.3) by one of the quadrature formulas.

Table 1

	ε				
	0.02	0.05	0.10	0.15	0.20
A_0	0.13279	0.16117	0.19403	0.21982	0.24252
$A_2\cdot10$	—0.68209	—0.73630	—0.72402	—0.70868	—0.68526
$A_4\cdot10^2$	—1.3115	0.083613	—0.84254	—0.50500	—0.20875
$A_6\cdot10^2$	—0.43721	—0.45794	—0.0035444	0.18313	0.28892
$A_8\cdot10^3$	—0.86645	—4.2548	1.4015	1.8660	1.7295
$A_{10}\cdot10^3$	—8.1259	0.97589	1.1747	0.91164	0.49437
$A_{12}\cdot10^3$	1.0691	0.33955	0.65397	0.24670	—0.0085908
$A_{14}\cdot10^4$	2.7912	1.0673	2.5336	—0.18771	—0.82858
$A_{16}\cdot10^4$	2.5506	4.3947	0.41799	—0.61749	—0.39833
$A_{18}\cdot10^4$	2.5964	2.3446	—0.34982	—0.36215	—0.11618
$A_{20}\cdot10^4$	1.6284	1.0846	—0.30902	—0.11260	—0.027086

Table 2

x/a	ε				
	0.02	0.05	0.10	0.15	0.20
0.00	0.481	0.496	0.474	0.475	0.474
0.10	0.471	0.497	0.474	0.475	0.474
0.20	0.450	0.497	0.473	0.475	0.474
0.30	0.440	0.493	0.474	0.475	0.476
0.40	0.474	0.481	0.476	0.475	0.471
0.50	0.483	0.473	0.478	0.475	0.470
0.60	0.502	0.439	0.478	0.474	0.468
0.70	0.490	0.431	0.481	0.473	0.464
0.80	0.479	0.456	0.485	0.470	0.457
0.85	0.497	0.481	0.484	0.466	0.456
0.90	0.553	0.522	0.488	0.477	0.475
0.95	0.632	0.588	0.524	0.539	0.562
0.975	0.712	0.703	0.650	0.697	0.747
0.99	0.874	0.986	0.963	1.061	1.152

We can approximately replace the infinite system (2.10) by a finite system of 11 equations with 11 unknowns. This finite system was solved several times in application to distinct values of the parameter ε. The results of the calculations are presented in Table 1.

Knowing the coefficients A_{2n}, we can easily find the quantities c and $p(x)$. Presented below are values of the coefficient γ (formula (3.2)) for some ε

$\varepsilon = 0.02$	0.05	0.10	0.15	0.20
$\gamma = 1.5260$	1.2573	1.0444	0.92184	0.83555

Tables of the Chebyshev polynomials [6] were used in calculating the function $p(x)$ by means of (3.3). Values of the quantity $aP^{-1}p(x)$ are presented in Table 2 for some ε and x/a.

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AXISYMMETRIC STRAIN OF AN ELASTIC LAYER WITH A CIRCULAR LINE OF SEPARATION OF THE BOUNDARY CONDITIONS ON BOTH FACES

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The problem of impressing a circular stamp into the upper face of a homogeneous elastic layer is considered. The layer rests on a stiff base weakened by a circular hole coaxial with the stamp and of the same radius. The surface of the stamp base possesses axial symmetry. The parts of the layer face outside the limits of contact are stress-free; there is no friction or cohesion between the layer and the stamp nor between the layer and the base.